Combinatorial Search, Second compulsory assignment

A *bilinear function* is a map $f : \mathbf{R}^n \to \mathbf{R}$ defined by

$$f(x_1, x_2, \ldots, x_n) = \sum_{i,j=1 (i \neq j)}^{n} \alpha_{i,j} x_i x_j + \sum_{i=1}^{n} \beta_i x_i$$

for real constants $\alpha_{i,j}, \beta_i$.

A *bilinear program* is an optimization instance of the following kind: Find $x \in F$ maximizing $f(x)$, where f is a bilinear function and F is a set of feasible solutions given by a set of linear inequalities.

Clearly, bilinear programs generalize linear programs. In general, they much harder to solve: Your task is to show that solving bilinear programs to optimality is **NP**-hard, even for the special case of $F = [0,1]^n$.

- First, show the following *integrality theorem* for bilinear programs with $F = [0,1]^n$: There is an integer-valued (i.e., {0,1}-valued) solution attaining the maximum value of f. **Hint:** What happens to a bilinear function if all variables except one are fixed to constants?

The decision problem L_{BLP} associated with bilinear programming over $F = [0,1]^n$ is the following: Given a bilinear program with rational coefficients $\alpha_{i,j}$ and β_i in the expression for f and a rational value v, is there a solution $x \in [0,1]^n$ so that $f(x) \geq v$?

- Show that L_{BLP} is in **NP**.
- Show that L_{BLP} is **NP**-complete. **Hint:** Reduce from MAX INDEPENDENT SET. The expression $\sum_{i \in V} x_i - |V| \sum_{(i,j) \in E} x_i x_j$ could be useful...